Find two linearly independent series solutions of $y'' + x^2 y' + xy = 0$ about $x = 0$. **SCORE: _____ / 13 PTS**

You must find the recurrence relation for the coefficients, as well as the first four non-zero terms of each solution.

$$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$y'' + x^2 y' + xy = \underbrace{\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}}_{(1)} + \underbrace{\sum_{n=1}^{\infty} n a_n x^{n+1}}_{(1)} + \underbrace{\sum_{n=0}^{\infty} a_n x^{n+1}}_{(1)}$$

SHIFT LIMITS DOWN 3

$$(1)\ \sum_{n=-1}^{\infty} (n+3)(n+2) a_{n+3} x^{n+1}$$

$$= 2(1)a_2 + 3(2)a_3 x + a_0 x$$

$$(1)\ + \sum_{n=1}^{\infty} [(n+3)(n+2) a_{n+3} + n a_n + a_n] x^{n+1} = 0$$

(1) $2a_2 = 0 \rightarrow a_2 = 0$

(1) $3(2)a_3 + a_0 = 0 \rightarrow a_3 = -\frac{1}{3 \cdot 2} a_0$

$$a_{n+3} = \frac{-(n+1)a_n}{(n+3)(n+2)} \quad (2)$$

IF $a_0 = 1, a_1 = 0$: $a_3 = -\frac{1}{3 \cdot 2} a_0 = -\frac{1}{3 \cdot 2}$

$a_4 = 0 = a_7 = a_{10} = a_{13} \cdots$

$a_5 = 0 = a_8 = a_{11} = a_{14} \cdots$

$a_6 = \frac{-4}{6 \cdot 5} a_3 = \frac{4 \cdot 1}{6 \cdot 5 \cdot 3 \cdot 2}$

$a_9 = \frac{-7}{9 \cdot 8} a_6 = -\frac{7 \cdot 4 \cdot 1}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2}$

$$y_1 = \underbrace{1}_{(\frac{1}{2})} - \underbrace{\frac{1}{3 \cdot 2} x^3}_{(\frac{1}{2})} + \underbrace{\frac{4 \cdot 1}{6 \cdot 5 \cdot 3 \cdot 2} x^6}_{(\frac{1}{2})} - \underbrace{\frac{7 \cdot 4 \cdot 1}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2} x^9}_{(\frac{1}{2})} + \cdots$$

$$= 1 - \frac{1}{3.2}x^3 + \frac{4^2\cdot 1^2}{6\cdot 5\cdot 4\cdot 3\cdot 2}x^6 - \frac{7^2\cdot 4^2\cdot 1^2}{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}x^9 + \cdots$$

$$= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\prod_{k=1}^{n}(3k-2)^2}{(3n)!} x^{3n}$$

$\frac{1}{2}$ BONUS

IF $a_0 = 0, a_1 = 1$: $a_3 = 0 = a_6 = a_9 = a_{12} \cdots$

$$a_4 = \frac{-2}{4.3}a_1 = \frac{-2}{4.3}$$

$$a_5 = 0 = a_8 = a_{11} = a_{14} \cdots$$

$$a_7 = \frac{-5}{7\cdot 6}a_4 = \frac{5\cdot 2}{7\cdot 6\cdot 4\cdot 3}$$

$$a_{10} = \frac{-8}{10\cdot 9}a_7 = -\frac{8\cdot 5\cdot 2}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}$$

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$$y_2 = x - \frac{2}{4.3}x^4 + \frac{5\cdot 2}{7\cdot 6\cdot 4\cdot 3}x^7 - \frac{8\cdot 5\cdot 2}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}x^{10} + \cdots$$

$$= x - \frac{2^2}{4\cdot 3\cdot 2}x^4 + \frac{5^2\cdot 2^2}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}x^7 - \frac{8^2\cdot 5^2\cdot 2^2}{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}x^{10} + \cdots$$

$$= x + \sum_{n=1}^{\infty} (-1)^n \frac{\prod_{k=1}^{n}(3k-1)^2}{(3n+1)!} x^{3n+1}$$

$\frac{1}{2}$ BONUS

Find two linearly independent series solutions of $2xy'' - y' + 2y = 0$ about $x = 0$. **SCORE: _____ / 17 PTS**

You must find the recurrence relation for the coefficients, as well as the first four non-zero terms of each solution.

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \qquad y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \qquad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$2xy'' - y' + 2y$ (1) (1) (1)

$$= \underbrace{\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1}}_{} - \underbrace{\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}}_{} + \underbrace{\sum_{n=0}^{\infty} 2a_n x^{n+r}}_{}$$

SHIFT LIMITS UP 1

(1) $\underbrace{\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}}_{}$

$$= 2r(r-1)a_0 x^{r-1} - ra_0 x^{r-1} \quad (1)$$

$$+ \underbrace{\sum_{n=1}^{\infty} \left[2(n+r)(n+r-1)a_n - (n+r)a_n + 2a_{n-1}\right] x^{n+r-1}}_{} = 0$$

$(2r(r-1) - r)a_0 = 0 \longrightarrow \underbrace{2r(r-1) - r = 0}_{}$ (2)

$r(2r-3) = 0$

$\underbrace{r = 0, \tfrac{3}{2}}_{}$ (1)

$r=0$: $\underbrace{2n(n-1)a_n - na_n + 2a_{n-1} = 0}_{}$ (1)

$n(2n-3)a_n + 2a_{n-1} = 0$

$$\underbrace{a_n = -\frac{2a_{n-1}}{n(2n-3)}}_{} \quad (2)$$

$$\text{IF } a_0 = 1,\ a_1 = -\frac{2}{1(-1)} a_0 = -\frac{2}{1(-1)}$$

$$a_2 = -\frac{2}{2(1)} a_1 = \frac{2^2}{2\cdot 1\cdot 1\cdot(-1)}$$

$$a_3 = -\frac{2}{3(3)} a_2 = -\frac{2^3}{3\cdot 2\cdot 1\cdot 3\cdot 1\cdot(-1)}$$

$$y_1 = \underbrace{1}_{(\frac{1}{2})} + \underbrace{2x}_{(\frac{1}{2})} - \underbrace{\frac{2^2}{2\cdot 1\cdot 1} x^2}_{(\frac{1}{2})} + \underbrace{\frac{2^3}{3\cdot 2\cdot 1\cdot 3\cdot 1} x^3}_{(\frac{1}{2})} - \cdots$$

$$= 1 + 2x - \frac{2^2}{2!\cdot 1}x^2 + \frac{2^3\cdot 2}{3!\ 3\cdot 2\cdot 1}x^3 - \frac{2^4\cdot 4\cdot 2}{4!\ 5\cdot 4\cdot 3\cdot 2\cdot 1}x^4 \dots$$

$$= 1 + 2x + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n\cdot 2^{n-2}(n-2)!}{n!(2n-3)!}x^n$$

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$$= 1 + 2x + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^{2n-2}(n-2)!}{n!(2n-3)!}x^n$$

(1/2) BONUS

$r=\frac{3}{2}$: $2(n+\frac{3}{2})(n+\frac{1}{2})a_n - (n+\frac{3}{2})a_n + 2a_{n-1} = 0$ (1)

$$(2n+3)(2n+1)a_n - (2n+3)a_n + 4a_{n-1} = 0$$

$$2n(2n+3)a_n + 4a_{n-1} = 0$$

$$a_n = -\frac{2}{n(2n+3)}a_{n-1}$$

IF $a_0 = 1$, $a_1 = \frac{-2}{1\cdot 5}a_0 = \frac{-2}{1\cdot 5}$

$$a_2 = \frac{-2}{2\cdot 7}a_1 = \frac{2^2}{2\cdot 1\cdot 7\cdot 5}$$

$$a_3 = \frac{-2}{3\cdot 9}a_2 = \frac{-2^3}{3\cdot 2\cdot 1\cdot 9\cdot 7\cdot 5}$$

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$$y_2 = x^{\frac{3}{2}}\left[1 - \frac{2}{1\cdot 5}x + \frac{2^2}{2\cdot 1\cdot 7\cdot 5}x^2 - \frac{2^3}{3\cdot 2\cdot 1\cdot 9\cdot 7\cdot 5}x^3 + \dots\right]$$

$$= x^{\frac{3}{2}}\left[1 - \frac{2}{1!5}x + \frac{2^2}{2!7\cdot 5}x^2 - \frac{2^3}{3!9\cdot 7\cdot 5}x^3 + \dots\right]$$

$$= x^{\frac{3}{2}}\left[1 + \sum_{n=1}^{\infty}(-1)^n\frac{2^n}{n!\prod_{k=1}^{n}(2k+3)}x^n\right]$$

(1/2) BONUS